

BESOV'S TYPE EMBEDDING THEOREM FOR BILATERAL GRAND LEBESGUE SPACES

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ABSTRACT.

In this paper we obtain the non - asymptotic norm estimations of Besov's type between the norms of a functions in different Bilateral Grand Lebesgue spaces (BGLS).

We also give some examples to show the sharpness of these inequalities.

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1. INTRODUCTION. STATEMENT OF PROBLEM. NOTATIONS. AUXILIARY FACTS.

A. Introduction.

Let X be the set $[0, 2\pi]$ endowed with the normalized Lebesgue measure

$$\mu(A) = (2\pi)^{-1} \int_A dx, \quad A \subset X.$$

Let also B be some rearrangement invariant (r.i.) space with the norm of a function $f : X \rightarrow R$ denoted by $\|f\|_B$ over the set X . We denote by $T(h)f$ for arbitrary function f from the space B the shift operator

$$T(h)f(x) = f(x + h), \quad h \in (-2\pi, 2\pi),$$

where all arithmetical operations over the arguments of a functions are understood modulo 2π ; and we introduce the correspondent modulo of continuity

$$\omega^{(B)}(f, \delta) \stackrel{\text{def}}{=} \sup_{h: |h| \leq \delta} \|T(h)f - f\|_B, \quad \delta \in X. \quad (1)$$

We denote as usually the classical $L(p)$, $p \geq 1$ Lebesgue norm

$$|f|_p = \left(\int_X |f(x)|^p d\mu \right)^{1/p}; \quad f \in L(p) \Leftrightarrow |f|_p < \infty.$$

We will denote for the spaces $L(p)$

$$\omega^{(L(p))}(f, \delta) = \omega(f, \delta)_p.$$

It follows from Hölder inequality that if $1 \leq p \leq q$, then $|f|_p \leq |f|_q$, or simple $L(q) \subset L(p)$.

Inversely, if $f \in L(p)$ and

$$\omega^{(L(p))}(f, \delta) = \omega(f, \delta)_p \leq C \delta^\alpha, \quad (2)$$

where $C, \alpha = \text{const} > 0$, $\alpha p < 1$, then for any $q \in (p, p/(1 - \alpha p))$

$$f \in L(q), \quad (3)$$

see, e.g. [5], chapter 4; [6], chapter 2.

Analogous assertions are true for the classical Besov's spaces [5], [6], [4], chapter 5, section 4, p. 332-346 etc; for the Sobolev's spaces [2], [8], [11], [25], [27], [35] [38], [39], [41] etc.

B. Statement of problem.

Our aim is a generalization of the estimation (2), (3) on the so-called Bilateral Grand Lebesgue Spaces $BGL = BGL(\psi) = G(\psi)$, i.e. when $f(\cdot) \in G(\psi)$ and to show the precision of obtained estimations by means of the constructions of suitable examples.

C. Another notations.

Let $f \in L(p)$, $p \in (1, \infty)$ and $n = 1, 2, 3, \dots$. We denote as ordinary the error of the best trigonometric approximation in the norm $|\cdot|_p$ of the function $f(\cdot)$ by the trigonometric polynomials of degree $\leq n$ by $E_n(f)_p$:

$$E_n(f)_p = \inf_{\deg(U) \leq n} |f - U|_p,$$

where $U = U(x)$ is arbitrary trigonometric polynomial of degree less or equal n : $U(\cdot) \in Q(n)$, where $Q(n)$ is the set of all trigonometric polynomial of degree less or equal n :

$$Q(n) = \{a_0/2 + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx))\}.$$

D. Auxiliary facts.

It follows from the classical *approximation theory* (generalized Jackson inequalities) that there exists an absolute constant $C \in (0, \infty)$ such that

$$E_n(f)_p \leq C \cdot \omega(f, \delta)_p, \quad (4)$$

see for example, [9], [10].

We recall also the well-known Nikol'skii inequality: for all trigonometric polynomial $U = U(x)$ of degree less or equal n : $\deg(U) \leq n$ and the values p, q : $1 \leq p \leq q \leq \infty$

$$|U|_q \leq 2 n^{1/p-1/q} |U|_p, \quad (5)$$

see [10], [30], [31], [32].

Note that the inequality (5) is trivially satisfied also in the case $1 \leq q < p$.

2. BILATERAL GRAND LEBESGUE SPACES

We recall briefly the definition and needed properties of these spaces. More details see in the works [15], [16], [19], [20], [33], [34], [23], [21], [22] etc. More about rearrangement invariant spaces see in the monographs [4], [24].

For a and b constants, $1 \leq a < b \leq \infty$, let $\psi = \psi(p)$, $p \in (a, b)$, be a continuous positive function such that there exists a limits (finite or not) $\psi(a+0)$ and $\psi(b-0)$, with conditions $\inf_{p \in (a,b)} \psi(p) > 0$ and $\min\{\psi(a+0), \psi(b-0)\} > 0$. We will denote the set of all these functions as $\Psi(a, b)$.

The Bilateral Grand Lebesgue Space (in notation BGLS) $G(\psi; a, b) = G(\psi)$ is the space of all measurable functions $f : X \rightarrow R$ endowed with the norm

$$\|f\|_{G(\psi)} \stackrel{def}{=} \sup_{p \in (a,b)} \left[\frac{|f|_p}{\psi(p)} \right], \quad (6)$$

if it is finite.

In the article [34] there are many examples of these spaces. For instance, in the case when $1 \leq a < b < \infty$, $\beta, \gamma \geq 0$ and

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = (p-a)^{-\beta} (b-p)^{-\gamma};$$

we will denote the correspondent $G(\psi)$ space by $G(a, b; \beta, \gamma)$; it is not trivial, non - reflexive, non - separable etc. In the case $b = \infty$ we need to take $\gamma < 0$ and define

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = (p-a)^{-\beta}, p \in (a, h);$$

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = p^{-\gamma} = p^{-|\gamma|}, p \geq h,$$

where the value h is the unique solution of a continuity equation

$$(h-a)^{-\beta} = h^{-\gamma}$$

in the set $h \in (a, \infty)$.

We will denote for simplicity $\Psi(1, b) = \Psi(b)$, $1 < b \leq \infty$; $\psi(1, b) = \psi(b)$; $\psi(1, b; \alpha, \beta) = \psi(b, \beta)$; the value α in this case is not essential.

The $G(\psi)$ spaces over some measurable space (X, F, μ) with condition $\mu(X) = 1$ (probabilistic case) appeared in an article [23].

The BGLS spaces are rearrangement invariant spaces and moreover interpolation spaces between the spaces $L_1(R^d)$ and $L_\infty(R^d)$ under real interpolation method [7], [21].

It was proved also that in this case each $G(\psi)$ space coincides with the so - called *exponential Orlicz space*, up to norm equivalence. In others quoted publications were investigated, for instance, their associate spaces, fundamental functions $\phi(G(\psi; a, b); \delta)$, Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.

Let $g : X \rightarrow R$ be some measurable function such that

$$g \in \cup_{p>1} L(p).$$

We can then introduce the non-trivial function $\psi_g(p)$ as follows:

$$\psi_g(p) \stackrel{def}{=} |g|_p. \quad (7)$$

This choosing of the function $\psi_g(\cdot)$ will be called *natural choosing*.

Remark 1. If we introduce the *discontinuous* function

$$\psi_r(p) = 1, p = r; \psi_r(p) = \infty, p \neq r, p, r \in (a, b) \quad (8)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$\|f\|_{G(\psi_r)} = |f|_r.$$

Thus, the Bilateral Grand Lebesgue spaces are the direct generalization of the classical exponential Orlicz's spaces and Lebesgue spaces L_r .

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [15], [19], theory of probability in Banach spaces [26], [23], [33], in the modern non-parametrical statistics, for example, in the so-called regression problem [33].

We use symbols $C(X, Y)$, $C(p, q; \psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X, Y)$ and $C_2(X, Y)$. The relation $g(\cdot) \asymp h(\cdot)$, $p \in (A, B)$, where $g = g(p)$, $h = h(p)$, $g, h : (A, B) \rightarrow R_+$, denotes as usually

$$0 < \inf_{p \in (A, B)} h(p)/g(p) \leq \sup_{p \in (A, B)} h(p)/g(p) < \infty.$$

The symbol \sim will denote usual equivalence in the limit sense.

The particular orderings in the set $G\Psi$ may be introduced as follows. We will write $\psi_1(\cdot) < \psi_2(\cdot)$, or equally $\psi_2(\cdot) > \psi_1(\cdot)$, if

$$\sup_{p \in (A, B)} \psi_1(p)/\psi_2(p) < \infty. \quad (9)$$

We will write $\psi_1(\cdot) \ll \psi_2(\cdot)$, or equally $\psi_2(\cdot) \gg \psi_1(\cdot)$, if

$$\lim_{\psi_2(p) \rightarrow \infty} \psi_1(p)/\psi_2(p) = 0. \quad (10)$$

We will denote as ordinary the indicator function

$$I(x \in A) = 1, x \in A, \quad I(x \in A) = 0, x \notin A;$$

here A is a measurable set.

3. MAIN RESULT: UPPER ESTIMATIONS FOR EMBEDDING THEOREM

Note at fist that if $\psi_1(\cdot) < \psi_2(\cdot)$, then $G(\psi_1)$ is closed subspace of the space $G(\psi_2)$: if $f \in G(\psi_1)$ and $\|f\|_{G(\psi_1)} = 1$, then

$$|f|_p \leq \psi_1(p) \leq \psi_2(p) \times \sup_p \frac{\psi_1(p)}{\psi_2(p)} \leq C \times \psi_2(p),$$

hence $\|f\|_{G(\psi_2)} \leq C$.

As a consequence: if $\psi_1(\cdot) < \psi_2(\cdot)$ and $\psi_2(\cdot) < \psi_1(\cdot)$, then the spaces $G(\psi_1)$ and $G(\psi_2)$ coincide up to norm equivalence.

It is known, see [34], that if $\psi_1(\cdot) \ll \psi_2(\cdot)$, then the space $G(\psi_1)$ is compact embedded in the space $G(\psi_2)$.

Further, we suppose $f(\cdot) \in G(\psi)$ for some $\psi(\cdot) \in \Psi(b)$. Let us introduce the set Z as a set of all strict monotonically increasing sequences of natural numbers $Z = \{n(k)\}$ such that

$$n(1) = 1, \quad n(k+1) \geq n(k) + 1, \quad k = 1, 2, \dots \quad (11)$$

For instance, the sequence $n_0(k) = 2^k - 1$ belongs to the set Z .

Let us define a new function (more exactly, a functional)

$$\nu(q, p) = \nu(\|f\|G(\psi), \omega^{(G(\psi))}(f, \cdot); \psi; q, p)$$

as follows:

$$\nu(q, p) = \inf_{\{n(\cdot) \in Z\}} \left\{ \|f\|G(\psi) + 32 \sum_{k=1}^{\infty} [n^{1/p-1/q}(k+1) \cdot \omega^{(G(\psi))}(f, 1/n(k))] \right\},$$

$$\theta(q) = \theta(q; \|f\|G(\psi), \omega^{(G(\psi))}(f, \cdot); \psi; q, p) = \inf_{p \in (1, b)} \nu(\|f\|G(\psi), \omega^{(G(\psi))}(f, \cdot); \psi; q, p). \quad (12)$$

Theorem 1. LET $\exists \psi(\cdot) \in \Psi(b)$ FOR WHICH $f(\cdot) \in G(\psi)$. ASSUME THAT FOR SOME $b_1 \geq b$ THE FUNCTION $\nu = \nu(q)$, $q \in (1, b_1)$ IS FINITE. THEN

$$|f|_q \leq 6\theta(q)$$

OR EQUALLY

$$\|f\|G(\theta) \leq 6. \quad (13)$$

Proof. We follow for the method offered by Il'in, see for example [6], chapter 4.

Let the function $f = f(x)$, $x \in X$ belongs to the space $G(\psi)$; so that therefore

$$|f|_p \leq \psi(p), \quad \sup_{h: |h| \leq \delta} |T(h)f - f|_p \leq \omega_p(f, \delta) \leq \psi(p) \omega^{(G(\psi))}(\delta), \quad p \in (1, b).$$

Let $n(k)$ be arbitrary sequence from the set Z and let also $m = m(n) = \text{Ent}(n/2)$, $n \geq 4$, where $\text{Ent}(z)$ denotes the integer part of a positive number z , $V_n[f](x) = V_n(x)$, $n = 2, 3, 4, \dots$ be a sequence of the classical trigonometrical Valee-Poussin polynomials of degree less than $2n$ for the function $f(\cdot)$. Recall that $V_n[f](x) =$

$$[2\pi m(n)]^{-1} \int_0^{2\pi} f(x+t) \sin[(2n+1-m(n))/2] \sin[0.5(m(n)+1)t] \sin^{-2}(t/2) dt. \quad (14)$$

It follows from the generalized Jackson inequalities [1], [12], [29], volume 1, chapter 4, p. 83-85, [40] that

$$|V_n[f]|_p \leq C_0 |f|_p,$$

$$|f - V_n[f]|_p \leq C \omega(f, 1/n)_p \leq C \psi(p) \omega^{(G(\psi))}(f, 1/n). \quad (15)$$

with some *absolute* constants C_0, C ; for example, it may be fetched $C_0 = 3$, $C = 12$, see [29], volume 1, chapter 4, p. 83-85.

Let us consider the following sequence of trigonometrical polynomials

$$P(x) = Q_{n(1)}(x); \quad Q_k(x) = V_{n(k+1)}[f](x) - V_{n(k)}[f](x); \quad (16)$$

then $|P(\cdot)| \leq 3\psi(p)\|f\|G(\psi)$ and $\deg Q_k \leq 2n(k+1)$;

$$|Q_k|_p \leq |f - V_{n(k+1)}[f]|_p + |f - V_{n(k)}[f]|_p \leq C [\omega(f, 1/n(k+1))_p + \omega(f, 1/n(k))_p] \leq$$

$$2C \omega(f, 1/n(k))_p \leq C_2 \psi(p) \omega^{(G(\psi))}(f, 1/n(k)), \quad (17)$$

and

$$f(x) = P(x) + \sum_{k=1}^{\infty} Q_k(x)$$

almost everywhere and in all the $L(p)$ sense for all the values p for which $\psi(p) < \infty$, i.e. for all the values p from the interval $p \in (1, b)$.

We conclude using the Nikol'skii inequality:

$$|P|_q \leq C_2 \psi(p) \|f\| G(\psi),$$

$$|Q_k|_q \leq C_3 [n(k+1)]^{1/p-1/q} \psi(p) \omega^{(G(\psi))}(f, 1/n(k)).$$

We get using the triangle inequality for the $L(q)$ norm:

$$|f|_q \leq C_2 \psi(p) \|f\| G(\psi) + C_3 \sum_{k=1}^{\infty} [n(k+1)]^{1/p-1/q} \psi(p) \omega^{(G(\psi))}(f, 1/n(k)). \quad (18)$$

Since the value p and the sequence $\{n(k)\}$ are arbitrary, we obtain tacking the infimum over p and $\{n(k)\}$ the assertion of theorem 1.

4. EXAMPLES.

1. If we choose $n(k) = 2^k - 1$, we state: $f(\cdot) \in G(\zeta)$, where

$$\zeta(q) = \inf_{p \in (1, b)} \left[\psi(p) + \sum_{k=1}^{\infty} 2^{k(1/p-1/q)} \psi(p) \omega^{(G(\psi))}(f, 2^{-k}) \right].$$

2. Further, suppose in addition to the conditions for last assertion that there is a constant $\alpha \in (0, 1/b)$ such that

$$\omega^{(G(\psi))}(f, \delta) \leq C_4 \delta^\alpha, \quad \delta \in [0, 1];$$

then

$$\zeta(q) = \inf_{p \in (1, b)} \frac{\psi(p)}{\alpha - (1/p - 1/q)}, \quad (19)$$

as long as

$$\sum_{k=1}^{\infty} 2^{-k(\alpha - (1/p - 1/q))} \sim (\alpha - (1/p - 1/q))^{-1},$$

when $\alpha \in (1/p - 1/q, 1/p - 1/q + 1)$, $\alpha < 1/b$.

Notice that the function $\zeta(q)$ is finite for all the values q from the open interval $q \in (1, b_1)$, $b_1 \stackrel{\text{def}}{=} b/(1 - \alpha b)$.

Remark 2. The assertion (19) is exact, for instance, for the $G(\psi_r)$ spaces.

3. Furthermore, if $\exists \alpha = \text{const} \in (0, 1/b)$, $\alpha_2 = \text{const} > -1$ such that

$$\omega^{(G(\psi))}(f, \delta) \leq C_5 \delta^\alpha |\log \delta|^{\alpha_2}, \quad \delta \in [0, 1/e],$$

then

$$\zeta(q) = \inf_{p \in (1, b)} \frac{\psi(p)}{[\alpha - (1/p - 1/q)]^{\alpha_2 + 1}}.$$

When $\alpha_2 = -1$, then

$$\zeta(q) = \inf_{p \in (1, b)} \{\psi(p) \cdot [|\log(\alpha - (1/p - 1/q))| + 1]\}.$$

Note that in the case $\alpha_2 < -1$ we obtain a trivial result: $\nu(q) \asymp \psi(q)$.

The last assertions follow from the following elementary fact.

Lemma 1. Let the value η belongs to the interval $(0, 1)$. Consider the series

$$W(\eta) = W_{\alpha_2}(\eta) = \sum_{k=1}^{\infty} 2^{-k\eta} k^{\alpha_2}, \quad \alpha_2 = \text{const}.$$

The following equalities are true:

$$\mathbf{A.} \alpha_2 > -1 \Rightarrow W_{\alpha_2}(\eta) \asymp \eta^{-1-\alpha_2}; \quad (20A)$$

$$\mathbf{B.} \alpha_2 = -1 \Rightarrow W_{\alpha_2}(\eta) \asymp [|\log(\eta)| + 1]; \quad (20B)$$

$$\mathbf{C.} \alpha_2 < -1 \Rightarrow \sup_{\eta \in (0, 1)} W_{\alpha_2}(\eta) < \infty. \quad (20C)$$

Note that the last case is trivial for us.

4. Suppose in addition $\exists b \in (1, \infty), \beta \geq 0$, such that

$$\psi(p) = (b - p)^{-\beta}, \quad p \in (1, b). \quad (21)$$

We will distinguish a two cases: $\alpha < 1/b$ and $\alpha = 1/b$.

A1. The possibility $\alpha < 1/b$ and $\alpha_2 < -1$.

Let us denote $b_1 = b/(1 - \alpha b)$.

We have after simple calculations for the values $q \in (1, b/(1 - \alpha b)) = (1, b_1)$

$$\begin{aligned} \nu(q) &\leq \zeta_\beta(q) \stackrel{\text{def}}{=} \inf_{p \in (1, b)} \frac{(b - p)^{-\beta}}{[\alpha - (1/p - 1/q)]^{\alpha_2 + 1}} \sim \\ &(b_1 - q)^{-\beta - \alpha_2 - 1}, \quad q \in (1, b_1). \end{aligned}$$

Thus, if $f(\cdot) \in G\Psi(b; \beta)$ and

$$\omega^{(G\Psi(b; \beta))}(\delta) \leq C \delta^\alpha |\log \delta|^{\alpha_2}, \quad \delta \in (0, 1/e),$$

then $f \in G\Psi(b_1, \beta + \alpha_2 + 1)$ and

$$\|f\| G\Psi(b_1, \beta + \alpha_2 + 1) \leq C(\alpha, \alpha_2, \beta) \|f\| G\Psi(b, \beta). \quad (22)$$

A2. The variant $\alpha < 1/b$, but $\alpha_2 = -1$.

Here

$$\nu(q) \leq C \zeta_{\beta, 1}(q) \stackrel{\text{def}}{=} \inf_{p \in (1, b)} [(b - p)^{-\beta}] \cdot [|\log(\alpha - (1/p - 1/q))| + 1] \sim$$

$$(b_1 - q)^{-\beta} \cdot [|\log(b_1 - q)| + 1], \quad q \in (1, b_1).$$

B1. Let now $\alpha = 1/b$ and $\alpha_2 > -1$.

Note that in the case if $\alpha = 1/b$ we conclude formally $b_1 = +\infty$.

We have using theorem 1:

$$\nu(q) \leq C(\alpha, \alpha_2, \beta) q^{\beta+\alpha_2+1}, \quad q \in (1, \infty).$$

B2. Finally, let $\alpha = 1/b$, $\alpha_2 = -1$. We obtain:

$$\nu(q) \leq C_1(\alpha, \beta) q^\beta \cdot [\log q + 1].$$

5. EXACTNESS OF THE UPPER BOUNDS

In this section we built some examples in order to illustrate the exactness of upper estimations.

Theorem 2. THE ESTIMATION OF THEOREM 1 IS IN GENERAL CASE THE BEST POSSIBLE.

In order to prove this assertion, we consider the following example. Let $X = (0, 1)$, $d\mu = dx$, $\Delta = \text{const} \in (0, 1)$, $\gamma = \text{const} > 1$, $b = \text{const} \in (1, 1/\Delta)$; we will suppose further $\gamma \rightarrow \infty$;

$$f_0(x) := x^{-\Delta} |\log x|^\gamma I(x \in (0, 1)). \quad (23)$$

We find by the direct calculations at $p \in (1, 1/\Delta)$:

$$|f_0|_p^p = \int_0^1 x^{-p\Delta} |\log x|^{\gamma p} dx = \frac{\Gamma(\gamma p + 1)}{(1 - p\Delta)^{\gamma p + 1}}, \quad (24)$$

$$|f_0|_p \asymp (q_0 - p)^{-\gamma-1/p}, \quad q_0 \stackrel{\text{def}}{=} 1/\Delta.$$

and we take the natural function

$$\psi(p) = |f_0|_p, \quad p \in (1, b);$$

then

$$\omega^{(G(\psi_0))}(f_0, \delta) \asymp \delta^{1/b-1/q_0} |\log \delta|^\gamma, \quad \delta \in (0, 1/e); \quad (25)$$

so that here

$$\alpha = 1/b - 1/q_0, \quad \alpha_2 = \gamma, \quad \beta = 0.$$

It follows from theorem 1 that $f_0 \in G\nu(q_0, 1/\Delta)$, where

$$\nu(q) \asymp (q_0 - q)^{-\gamma-1}, \quad q \in (1, q_0) = (1, 1/\Delta);$$

but really

$$\nu_0(q) \stackrel{\text{def}}{=} |f_0|_q \asymp (q_0 - q)^{-\gamma-\Delta}.$$

We can see that the *extremal value* of the function ν_0 , i.e. the value

$$q_{\max} \stackrel{\text{def}}{=} \sup\{q, \nu_0(q) < \infty\} \quad (26)$$

is calculated exactly by means of theorem 1: $q_{\max} = q_0 = 1/\Delta$.

The *power* of the function ν_0 , i.e. the value

$$w(\nu_0) \stackrel{\text{def}}{=} \overline{\lim}_{q \rightarrow q_0 - 0} \frac{|\log \nu_0(q)|}{|\log(q - q_0)|} \quad (27)$$

evaluating by means of theorem 1 is equal to $\gamma + 1$, but really this value is equal to $\gamma + \Delta$.

Notice that

$$\lim_{\gamma \rightarrow \infty} \frac{\gamma + \Delta}{\gamma + 1} = 1. \quad (28)$$

6. CONCLUDING REMARK: COMPACTNESS OR NOT COMPACTNESS OF EMBEDDING OPERATOR

Let $\psi(\cdot)$ and $\theta(\cdot)$ be at the same as in theorem 1. Note that in general case the unit embedding operator $T : G(\psi) \in G(\theta)$ is not compact [34].

But if we consider the unit operator $T : G(\psi) \in G(\xi)$, where $\xi(\cdot) \ll \psi(\cdot)$, then this operator T is compact.

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